EVOLUTION OF NONLINEAR WAVES IN A DISSIPATIVE MEDIUM WITH DISPERSION

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The process of propagation of long-wave disturbances is modeled numerically in the article within the framework of the Burgers–Korteweg–de Vries equation, allowing for low-frequency absorption. Possible isomorphous solutions are examined.

The propagation of long-wave disturbances of final amplitude U in a medium having cubic dispersion is described by the Korteweg-de Vries equation [1-3]

$$U_t + UU_x + \beta U_{xxx} = 0 \tag{1}$$

If there is dissipation in the medium, then along with (1) we have

$$U_t + UU_x + \beta U_{xxx} - \alpha U_{xx} + \gamma U = 0 \tag{2}$$

The last two terms in (2) determine the high-frequency and low-frequency absorption. Equation (2) and modifications of it are obtained in an examination of the evolution of disturbances in a plasma, at the surface of a liquid, in a biphasic medium, in a medium containing heterogeneities, etc. [1-7].

When dispersion and high-frequency absorption are absent ($\beta = \alpha = 0$), we have from (2)

$$U_t + UU_x + \gamma U = 0 \tag{3}$$

Let us obtain an isomorphous solution for this equation. After the substitutions

$$\xi = x \exp(\gamma t), \quad U = \exp(\gamma t)\,\psi(\xi)$$

Eq. (3) takes the form

$$2\psi + \xi\psi_\xi + \mu\psi\psi_\xi = 0, \quad \mu = \exp(2\gamma t)\,\gamma^{-1} \tag{4}$$

Let us introduce $\varphi = \mu\psi$. Then from (4) we obtain

$$2\varphi + (\xi + \varphi)\,\varphi_\xi = 0 \tag{5}$$

This is an Abel equation of the second kind, which involves only isomorphous variables.

If we exclude the case of $\varphi \equiv 0$, we obtain for the function $\xi = \xi(\varphi)$ the equation

$$\frac{d\xi}{d\varphi} + \frac{1}{2\varphi}\xi = -\frac{1}{2} \tag{6}$$

Its solution is

$$\xi = C\varphi^{-1/2} - \varphi/3 \tag{7}$$

where C is a constant of integration.

From (7) one can find φ. Let $\beta = \gamma = 0$. Then (2) is converted into the well-known Burgers equation

$$U_t + UU_x - \alpha U_{xx} - 0 \tag{8}$$

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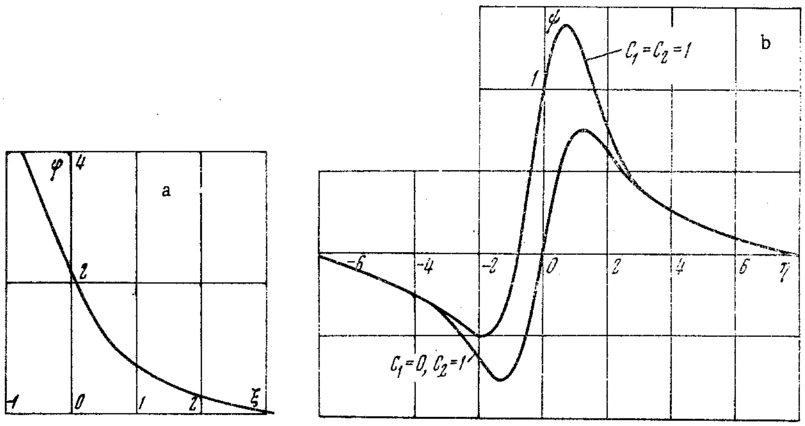

Fig. 1

It is easy to show that (8) has an isomorphous solution of the form

$$U(x,\ t) = \alpha\,(2\,\alpha t)^{-1/2}\,\psi\,[x\,(2\,\alpha t)^{-1/2}] \tag{9}$$

Substituting (9) into (8) we obtain an equation for

$$\frac{d^2\psi}{d\eta^2} - \psi\frac{d\psi}{d\eta} + \eta\frac{d\psi}{d\eta} + \psi = 0, \quad \eta = x\,(2\alpha t)^{-1/2} \tag{10}$$

When the amplitude of the disturbance is small enough and the nonlinear term in (10) can be neglected we have

$$\psi = e^{-\eta^2/2}\left(C_1 + C_2\int e^{-\eta^2/2}d\eta\right) \tag{11}$$

where C_1 and C_2 are constants of integration.

In the general case (10) the substitution

$$y = (\eta - \psi)\,/\,2 \tag{12}$$

leads to the Rikkat equation

$$y' + y^2 = 1/2 + \eta^2\,/\,4 + C \tag{13}$$

The substitution

$$z = \exp\,(\textstyle\int y\,d\eta) \tag{14}$$

gives

$$z'' = (1/2 + \eta^2\,/\,4 + C)\,z \tag{15}$$

If $\psi = \psi' = 0$ at $\eta = 0$, it follows from (12) and (13) that $C = -1/2$. Then

$$z = A_1\sqrt{\eta}\,J_{1/4}\,(i\eta^2\,/\,4) + A_2\sqrt{\eta}\,Y_{1/4}\,(i\eta^2\,/\,4) \tag{16}$$

where A_1 and A_2 are determined from the initial and boundary conditions and $J_{1/4}$ and $Y_{1/4}$ are Bessel functions.

If the initial conditions are such that the constant C in (15) is zero, then

$$z = e^{\eta^2/4}\left(B_1 + B_2\int e^{-\eta^2/4}d\eta\right) \tag{17}$$

Knowing z, one can determine ψ using (12) and (14). Equation (15) is not integrated in quadratures for other values of C.

Let us clarify the physical meaning of the isomorphous solution of Eq. (8). Suppose a distribution with a characteristic amplitude U_0 and a characteristic dimension λ is given as the initial condition for (8). Let us introduce the dimensionless values

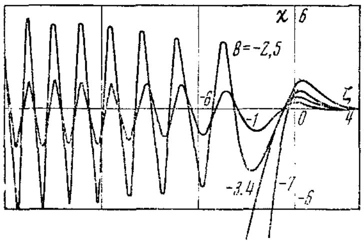

Fig. 2

$$\tau = U_0 t / \lambda,\ y = x / \lambda,\ v = U / U_0$$

From (8) we obtain

$$v_\tau + v v_y - \nu^{-1} v_{yy} = 0,\ \nu = U_0 \lambda / \alpha$$

Let us consider a succession of initial disturbances for which $\lambda \to 0$ but the quantity $U_0\lambda$ remains constant. Then for an identical value of α the solutions of Eq. (8) should be similar, since ν =const; hence, the limiting solution can contain only a combination of parameters having the dimensionality of length and velocity. In the Burgers equation this parameter is α. Then the limiting solution has the form of (9). It can be shown that the initial disturbance leading to (9) is

$$U(x, 0) = \alpha\nu\delta(x)$$

where $\delta(x)$ is the delta function. Actually, if A is an arbitrary constant, then using one representation of $\delta(x)$ we have

$$A\delta(x) = \lim_{\lambda \to 0} (A\lambda\pi^{-1} / (x^2 + \lambda^2))$$

Here A/λ is the characteristic velocity U_0, where $U_0\lambda = A =$ const. But $U_0\lambda = \alpha\nu$, so that $A = \alpha\nu$. Thus, the physical meaning of the isomorphous solution of (8) of the Burgers equation consists in a description of the propagation of initial disturbances of the type

$$U(x, 0) = \frac{\alpha\nu\lambda}{\pi(x^2 + \lambda^2)} \quad \text{at} \quad x \gg \lambda,\ t^{1/2} \gg l\alpha^{-1/2}$$

The physical meaning of the isomorphous solution of the Korteweg–de Vries equation, which is discussed below, is established in a similar way in [2]. The isomorphous solution $\varphi(\xi)$ of Eq. (5) and the solution (11) of Eq. (10) are illustrated in Fig. 1a and b, respectively. It is seen from Fig. 1a that $\varphi(\xi)$ has meaning when $\xi \gg 1$, i.e., $x \gg \lambda$ and $t \gg \gamma^{-1}$.

When there is no dissipation in the medium the propagation of the disturbance is described by the Korteweg–de Vries equation (1). It is shown in [2] that Eq. (1) is satisfied by the following isomorphous solution:

$$U(x, t) = \beta(3\beta t)^{-2/3}\chi[x(3\beta t)^{-1/3}] \tag{18}$$

Substituting (18) into (1) gives an equation for

$$\chi'' - \zeta\chi' + \chi\chi' - 2\chi = 0,\quad \zeta = x(3\beta t)^{-1/3} \tag{19}$$

If the damping as $\zeta \to \infty$ of the solution (19) is considered as exponential, then

$$\chi(\zeta) \approx B\, d\mathrm{Ai}(\zeta)/d\zeta \quad \text{as} \quad \zeta \to \infty \tag{20}$$

where $\mathrm{Ai}(\zeta)$ is the Airy function and B is a constant.

The behavior of $\chi(\zeta)$ at negative and small positive values of ζ is found in [2] by numerical integration of (19) at positive values of B. The behavior of $\chi(\zeta)$ at negative values of B is also clarified in this article (Fig. 2). The solutions for $B < 0$ and $B > 0$ differ markedly in the vicinity of the origin of coordinates. For $B < B_*$ the function $\chi(\zeta)$ for negative ζ oscillates with slowly-increasing amplitude. For $B > B_*$ the solutions χ have a singularity. As numerical experiments show,

$$B_*^{(+)} = -B_*^{(-)} = 3.35$$

Let us move to an examination of the nonstationary solutions of Eq. (2).

In [6, 7] corrections are found to the stationary solution of Eq. (1), when the dissipative terms in (2) are small and the wave obtained is close to cnoidal. Using numerical integration a solution is found in the present work for Eq. (2) for different ratios between the dispersion coefficient β and the absorption parameters α and γ.

The Gaussian distribution

$$U(x, 0) = a_0 \exp[-(x - x_0)^2 / l^2] \tag{21}$$

was set as an initial condition for (2).

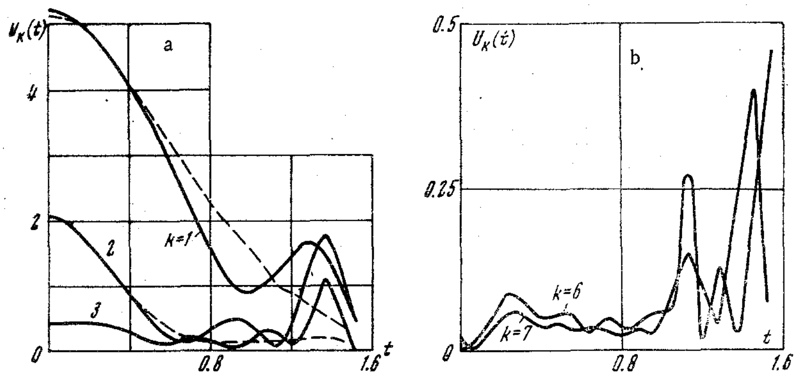

Fig. 3

The following periodic boundary conditions were considered:

$$U(x,\ t) = U(x + L,\ t) \tag{22}$$

To clarify the problem of the structure of the spectrum of the disturbance, the solution of Eq. (2) obtained was expanded in a Fourier series

$$U(x,\ t) = \sum_{k=1}^{N} U_k \exp(2\pi kx / L) \tag{23}$$

The number N of harmonics was chosen in such a way that the wavelength $\lambda_N = L/2\pi N$ of the N-th harmonic was many times greater than the spatial step in integration.

To test the correctness of the calculations a solitary stationary solution of the Korteweg–de Vries equation [1–3] was set up as the initial distribution for Eq. (2) at $\alpha = \gamma = 0$:

$$U(x,\ 0) = a\ \mathrm{sech}^2\,(a/12\,\beta)^{1/2}\,(x - x_0) \tag{24}$$

Its variation with time did not exceed the error in integrating Eq. (2). The accuracy of the computation was controlled by conservation laws. At $\alpha = \gamma = 0$ the conservation of the first two invariants of Eq. (1) was tested:

$$J_1 = \int_0^L U\,dx, \qquad J_2 = \int_0^L U^2\,dx$$

The greatest relative errors in the course of the calculations were as follows:

$$\Delta J_1 / J_1 \approx 3\cdot 10^{-6},\ \Delta J_2 / J_2 \approx 5\cdot 10^{-3}$$

The law of variation of the wave impulse with time was checked in the general case ($\alpha \neq 0,\ \gamma \neq 0$):

$$\int_0^L U(x,\ t)\,dx = \exp(-\gamma t)\int_0^L U(x,\ 0)\,dx$$

The dispersion parameter β, the amplitude a_0, and the width l of the initial distribution (21) were chosen in such a way that the similarity parameter $\sigma = (a_0 l^2/\beta)^{1/2}$ was both greater and smaller than $\sigma_* = \sqrt{12}$ [2].

In the case of very large σ ($\sigma \gg \sigma_*$) a set of solitons [2, 3] is formed. The dependence of the square of the Fourier component U_k^2 of the solution on time at $\sigma^2 = 7.5\cdot 10^5$, $\alpha = 10^{-4}$, and $\gamma = 0$ is shown in Fig. 3a, b. It is seen that the relative contribution of the first harmonics (k = 1, 2, 3) decreases with an increase in t, while that of the higher harmonics (k = 6, 7) increases. The same thing happens at $\sigma^2 = 9$. The behavior of $U_k^2(t)$ in the case of a conservative medium ($\alpha = \gamma = 0$) is shown by dashed lines.

Thus, high-frequency absorption "destabilizes" the wave of excitation. This has been observed experimentally, for example, in the propagation of sound in a biphasic medium consisting of a liquid and gas bubbles [5]. The situation is qualitatively the same for positive dispersion also. Destabilization of the wave can be interpreted qualitatively using the stationary solution of Eq. (2).

Let us shift to the variable $r = x - Vt$, setting $\gamma = 0$ and integrating the equation obtained once with the condition $U = U' = U''$ as $x \to \infty$, obtaining

$$\beta U'' - \alpha U' + U^2 / 2 - VU = 0 \tag{25}$$

It follows from an analysis of this equation [2] that if $\alpha > \alpha_* = (4\beta V)^{1/2}$, the profile of the wave becomes oscillatory. This can be generalized qualitatively to the case when the wave differs little from a stationary wave.

The numerical experiments show that the frequency of the oscillations increases in the process of evolution. This occurs mainly in the region where $U > 0$. Because of the law of conservation of an impulse (at $\gamma = 0$) the area of the initial disturbance is conserved. Therefore, the amplitude of the oscillations grows with a decrease in their wavelength. The role of the nonlinear effects intensifies with an increase in amplitude, which leads to further driving of the high-frequency oscillations.

The computation was conducted to the point where the wavelength of the oscillations was much greater than the step in integration with respect to x. Generally speaking, Eq. (2) becomes inapplicable on the appearance of short-wave oscillations. Therefore, the examination of the evolution of high-frequency oscillations has a qualitative character for the most part.

In the presence of low-frequency absorption the solutions of Eq. (2) become smoothed out. If $\alpha = 0$ and the amplitude of the disturbance is small enough that nonlinear effects can be neglected, it is seen from (2) that $U \sim \exp(-\gamma t)$. When $\alpha \neq 0$, the term containing γ in Eq. (2) stops or at least slows the destabilization of the wave depending on the relationship between the quantities α, β, and γ. This is confirmed by numerical experiments. We note that the effect of destabilization of a monochromatic wave in a nonlinear medium having high-frequency absorption is analogous to "dissipative" instability of a wave in dielectrics and in a plasma [8, 9].

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